Announcements

1) HW # 2 due on Thursday

2) Candidate talk 3 00 in CB 2070 (Math biology)

A much better proof that $|(0,1)| = \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x} - 2, & 0 < x \leq \frac{1}{2} \\ -\left(\frac{1}{-x+1} - 2\right), & \frac{1}{2} < x < 1 \end{cases}$$

is the required bijection from $(0,1)$ to $\mathbb{R}$.

Theorem: $|2^X| > |X|$ $\forall$ sets X.

Note: (bigger cardinality) this means there is no bijection from X to 2^X, but there is an injection.

proof If $|X| < \infty$, we did this last class. Assume the cardinality of X is infinite.

Let $f: X \to 2^X$ be any function. We will show that f is not surjective.

Define $A = \{x \in X : x \notin f(x)\}$.

Either

$\underline{A = \emptyset}$ If this is the case, then for all $x \in X$, $x \in f(x)$. This implies that $f(x) \neq \emptyset$ for any $x \in X$.

In this case, the empty set is not in the range of f, so f is not surjective.

$A \neq \phi$ Then $\exists\ y \in X$,

$$y \in A = \{x \in X : x \notin f(x)\}.$$

Suppose $\exists$ a $z \in X$, $\boxed{f(z) = A}$. Then we have one of two scenarios:

$z \in A$ immediate contradiction since $A = \{x \in X : x \notin f(x)\} = f(z)$ so $z \notin f(z) = A$

$z \notin A$ Then $z \notin f(z)$

$\Rightarrow$ $z \in A$ since

$A = \{x \in X : x \notin f(x)\}$

another contradiction.

Then there is no $z \in X$

with $f(z) = A$, again

implies that f is not

surjective.

Note: There is an injection

$g : X \to 2^X$, $g(x) = \{x\}$

Therefore, $|2^X| > |X|$

Since $\exists$ an injection from X to 2^X, but there can exist no bijection. $\square$

So, for example, $|2^{\mathbb{N}}| > |\mathbb{N}|$ and so $2^{\mathbb{N}}$ is not countable.

What is $|2^{\mathbb{N}}|$?

$$|2^{\mathbb{N}}| = |\mathbb{R}|.$$

(extra credit)

Chapter 2

Sequences and Series

Familiar from Calc II,

we will be focused on

proving the theorems

used in that class

(ratio test, monotone

convergence, etc.)

Definition: (Sequence)

A sequence is a function from the natural numbers to an arbitrary set X.

In this, we'll usually take

$$X = \mathbb{R}.$$

Examples.

1) $f(n) = \frac{1}{n!} \quad \forall n \in \mathbb{N}$

2) $f(n) = 0 \quad \forall n \in \mathbb{N}$

3) $f(n) = \frac{(-1)^n}{n} \quad \forall n \in \mathbb{N}$

Notation Instead of $f(n)$, we usually write a_n or b_n for the value of the function f at the integer n

Focus: Does a given sequence "converge"? If so, to what value?

Definition: (convergence)

A sequence $(a_n)_{n \in \mathbb{N}}$ is called convergent if $\exists$ $L \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ with

$$\boxed{|a_n - L| < \varepsilon}$$

$\forall$ $n \geq N$

Alternate Definition: $(a_n)_{n \in \mathbb{N}}$

converges to L if for every open interval containing L, all but finitely many of the a_n's are in the interval.

Example 1: $\left(\frac{1}{n!}\right)_{n\in\mathbb{N}}$

Find L. Claim: $L=0$

proof. Let $\varepsilon>0$. Then

$\frac{1}{\varepsilon}$ is a positive real number

By the Archimedean Property,

$\exists\ N\in\mathbb{N}$, $\frac{1}{\varepsilon}<N$.

Note: $N\leq(N!)$

So then $\frac{1}{\varepsilon}<(N!)$

For any $n \geq N$,

$n! \geq N!$, so $\frac{1}{n!} \leq \frac{1}{N!}$.

Hence, for all $n \geq N$,

$$
\begin{aligned}
|a_n - 0| &= |a_n| \\
&= \frac{1}{n!} \\
&\leq \frac{1}{N!} \\
&< \varepsilon. \quad \square
\end{aligned}
$$

Example 2 (a divergent sequence)

A sequence is divergent if it does not converge to any real number.

Example: $a_n = \dfrac{(-1)^n + 1}{2}$

$a_1 = 0$

$a_2 = 1$

$a_3 = 0$

$a_4 = 1$

$\vdots$